

# Generic hyperelliptic Prym varieties in a generalized Hénon–Heiles system\*

V.Z. Enolski,

School of Mathematics, University of Edinburgh, Edinburgh.

On leave from Institute of Magnetism, National Academy of Sciences of Ukraine,  
Kiev, Viktor.Enolskiy@ed.ac.uk,

Yu.N.Fedorov

Department of Mathematics I, Politechnic university of Catalonia, Barcelona,

Yuri.Fedorov@upc.edu,

A.N.W. Hone,

School of Mathematics, Statistics & Actuarial Science, University of Kent,  
Canterbury, United Kingdom

A.N.W.Hone@kent.ac.uk

## Abstract

It is known that the Jacobian of an algebraic curve which is a 2-fold covering of a hyperelliptic curve ramified at two points contains a hyperelliptic Prym variety. Its explicit algebraic description is applied to some of the integrable Hénon–Heiles systems with a non-polynomial potential. Namely, we identify the generic complex invariant manifolds of the systems as a hyperelliptic Prym subvariety of the Jacobian of the spectral curve of the corresponding Lax representation.

The exact discretization of the system is described as a translation on the Prym variety.

## 1 Introduction

Many algebraic completely integrable systems possess matrix Lax representations whose spectral curves admit symmetries (in particular, involutions). The Jacobians of the curves contain Abelian (Prym) subvarieties whose open subsets are identified with the complex invariant manifolds of the systems.

In most cases the Prym subvariety itself is not a Jacobian variety, in particular, due to the fact that it is not principally polarized. In low dimensions the subvariety can be related to the Jacobian of an algebraic curve via an isogeny, and the latter curve also appears as the curve of separation of variables for the system (see [1, 16], for example).

---

\*AMS Subject Classification 14H70, 14H40, 70H06, 37J35

On the other hand, as we know from [20, 7], when the spectral curve  $\tilde{C}$  admits an involution  $\sigma$  with two fixed points, and it covers a hyperelliptic curve, say  $C$ , the corresponding Prym variety becomes the Jacobian of (another) hyperelliptic curve  $C'$ , and thus can be referred to as a *hyperelliptic Prym variety*. Such a situation occurs, in particular, in finite-dimensional reductions (stationary flows) of the Sawada–Kotera and the Kaup–Kupershmidt hierarchies of PDEs (see [28]) and the associated integrable dynamical systems, such as the Hénon–Heiles systems described in [14, 15].

The special case where the two branch points of the covering (the images of the two fixed points of the involution  $\sigma$ ) are also related to one another by the hyperelliptic involution on  $C$  was considered in detail by several authors, see e.g., [11, 25, 13]. In that case, the covering curve  $\tilde{C}$  is also hyperelliptic, and the equation of the second curve  $C'$  representing the Prym variety is derived in a straightforward way. The corresponding applications to integrable systems were considered in [12, 13, 18, 23, 24].

For the general case,  $\tilde{C}$  is not hyperelliptic, and an algorithm for calculating the second curve  $C'$  was given only recently in [19]. In the present paper we apply a modification of the latter method to describe the complex invariant manifolds of certain integrable generalizations of the Hénon–Heiles system. To be precise, it is shown that the invariant manifolds of the cases (i) and (iii) of these systems are the 2-dimensional Prym subvariety of the Jacobian of a trigonal spectral curve  $\tilde{C}$ , and the second curve  $C'$  is identified with the curve associated with the separation of variables found previously in [6, 26]. Moreover, for an exact discretization (Bäcklund transformation)  $\mathcal{B}$  of the above systems constructed in [5], we describe each branch of  $\mathcal{B}$  as a translation on the Prym variety.

## 2 Double cover of a hyperelliptic curve with two branch points

Consider a hyperelliptic genus  $g$  curve  $C$ :  $y^2 = f(x)$ , where  $f(x)$  is a polynomial of degree  $2g + 1$  with simple roots. Any 2-fold covering of  $C$  ramified at two finite points  $P = (x_P, y_P), Q = (x_Q, y_Q) \in C$  (which are not related to each other by the hyperelliptic involution on  $C$ ) can be written in the form<sup>1</sup>

$$\tilde{C} : z^2 = y + h(x), \quad y^2 = f(x),$$

where  $h(x)$  is a polynomial of degree  $g + 1$  such that

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x)$$

with  $\rho(x)$  being a polynomial<sup>2</sup> of degree  $g$ . Thus  $\tilde{C}$  admits the involution  $\sigma : (x, y, z) \mapsto (x, y, -z)$ , with fixed points  $(x_P, y_P, 0), (x_Q, y_Q, 0) \in \tilde{C}$ . Then the genus of  $\tilde{C}$  is  $2g$  and the following was shown by D. Mumford and S. Dalaljan [20, 7]:

- 1) The Jacobian of  $\tilde{C}$  contains two  $g$ -dimensional Abelian subvarieties:  $\text{Jac}(C)$  and the Prym subvariety  $\text{Prym}(\tilde{C}, \sigma)$ . The former is invariant with respect to the involution  $\sigma$  extended to  $\text{Jac}(\tilde{C})$ , whereas the latter is anti-invariant.

---

<sup>1</sup>Here and below we identify a curve with its regularization.

<sup>2</sup>Here  $x_P$  or  $x_Q$  may or may not coincide with roots of  $\rho(x)$ .

2)  $\text{Prym}(\tilde{C}, \sigma)$  is a principally polarized Abelian variety and, moreover, is the Jacobian of a hyperelliptic curve  $C'$ .

It was further shown recently by A. Levin [19] that the second curve  $C'$  can be written explicitly as

$$w^2 = h(x) + Z, \quad Z^2 = h^2(x) - f(x) \equiv (x - x_P)(x - x_Q)\rho^2(x), \quad (1)$$

which is equivalent to the plane curve

$$[w^2 - h(x)]^2 = h^2(x) - f(x) \implies w^4 - 2h(x)w^2 + f(x) = 0.$$

The latter can be transformed to a standard hyperelliptic form which is given in [19].

Note that when the polynomial  $f(x)$  is of even degree  $2g + 2$  and  $g$  is odd ( $g = 1, 3, 5, 7, \dots$ ), the above formulas are still valid. However, when  $g$  is even, a different result holds, which is described as follows.

**Theorem 1.** (a) *In the case when the polynomial  $f(x)$  has even degree  $2g + 2$  and  $g = 2, 4, 6, \dots$ , any covering  $\tilde{C} \rightarrow C$  ramified at 2 finite points  $P = (x_P, y_P), Q = (x_Q, y_Q) \in C$  can be written in the form*

$$\left\{ y^2 = f(x), \quad z^2 = \frac{y + h(x)}{x - x_P} \quad \text{or, equivalently,} \quad z^2 = \frac{y + h(x)}{x - x_Q} \right\}, \quad (2)$$

where,  $h(x)$  is of degree at most  $g + 1$  and such that

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x), \quad (3)$$

for some polynomial  $\rho(x)$ .

(b) *The corresponding Prym variety is isomorphic to the Jacobian of a second genus  $g$  hyperelliptic curve  $C'$ , which can be written in the form*

$$\left\{ Y^2 = h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x), \quad w^2 = \frac{Y + h(x)}{x - x_P} \quad \text{or, equivalently,} \quad w^2 = \frac{Y + h(x)}{x - x_Q} \right\}. \quad (4)$$

The latter is transformed to the standard hyperelliptic form  $v^2 = P_{2g+2}(u)$ , where  $P_{2g+2}$  is a polynomial of degree  $2g + 2$ , by the birational transformation

$$x = \frac{x_Q u^2 - x_P}{u^2 - 1}, \quad w = \left( \frac{x_Q - x_P}{u^2 - 1} \right)^{g/2} v, \quad (5)$$

with inverse

$$v = \frac{w}{(x - x_Q)^{g/2}}, \quad u = \frac{(x - x_Q)w^2 - h(x)}{\rho(x)(x - x_Q)}.$$

(c) In the particular case  $g = 2$ , upon setting  $\rho(x) = Ax^2 + Bx + C$ , the hyperelliptic form of  $C'$  has the following structure

$$\begin{aligned} v^2 = & \frac{1}{(x_P - x_Q)^3} [h(x_Q)u^6 + (x_P - x_Q)\rho(x_Q)u^5 - [(x_P - x_Q)h'(x_Q) + 3h(x_Q)]u^4 \\ & + (x_P - x_Q)[\rho(x_P) + \rho(x_Q) - \rho_2(x_P - x_Q)^2]u^3 \\ & - [(x_P - x_Q)h'(x_P) - 3h(x_P)]u^2 + (x_Q - x_P)\rho(x_P)u - h(x_P)], \end{aligned} \quad (6)$$

where  $h'(x)$  is the derivative of  $h(x)$ .

**Remark.** Note that the equation (6) is symmetric with respect to  $x_P, x_Q$  in the sense that once these two values are interchanged, one obtains the equation of a birationally equivalent curve.

*Proof of Theorem 1.* We first notice that the function  $y + h(x)$  has poles of order  $g + 1$  at  $\infty_1, \infty_2$ , the two points at infinity on  $C$ . Then, in view of (3), the function  $z^2 = (y + h(x))/(x - x_P)$  has 2 simple zeros at some finite points  $P, Q$ , some double zeros elsewhere, and poles of even degree  $g$  at  $\infty_1$  and  $\infty_2$ . Thus the regularization of (2) is ramified over  $C$  at  $P, Q$  only.

The proof of the first and second items follows the same lines as that in the paper [19] for the case of  $f(x)$  having odd degree, and it uses the tower of curves constructed in [7]. We only add that the equivalence of the two alternative forms of the curves in (4) follows from the relation

$$\begin{aligned} \frac{Y + h(x)}{x - x_P} \left( \frac{Y}{(x - x_Q)\rho(x)} \right)^2 &= \frac{Y + h(x)}{x - x_P} \frac{h^2(x) - f(x)}{[(x - x_Q)\rho(x)]^2} \\ &= \frac{Y + h(x)}{x - x_P} \frac{(x - x_P)(x - x_Q)\rho^2(x)}{[(x - x_Q)\rho(x)]^2} = \frac{Y + h(x)}{x - x_Q}, \end{aligned}$$

where we used (3). Thus  $\frac{Y+h(x)}{x-x_P}, \frac{Y+h(x)}{x-x_Q}$  differ by a square factor, hence the two forms of (4) are transformed to each other by change  $w \rightarrow wY/[(x - x_Q)\rho(x)]$ . A similar argument gives the equivalence of the two forms of (2).

The substitution (5) was first suggested also in [19]; applying it to the second version of (4), one obtains

$$v^2 = \frac{(x - x_Q)\rho(x)u + h\left(\frac{x_Q u^2 - x_P}{u^2 - 1}\right)}{(x - x_Q)^{g+1}} = \frac{\frac{x_Q - x_P}{1 - u^2} \rho\left(\frac{x_Q u^2 - x_P}{u^2 - 1}\right) u + h\left(\frac{x_Q u^2 - x_P}{u^2 - 1}\right)}{\frac{(x_P - x_Q)^{g+1}}{(1 - u^2)^{g+1}}},$$

which, after simplification, gives a polynomial in  $u$  of degree  $2g + 2$  in the right hand side. In the case  $g = 2$  this gives the equation (6).  $\square$

**Remark.** The special case of the 2-fold covering  $\tilde{C} \rightarrow C$  when the branch points  $P = (x_P, y_P), Q = (x_Q, y_Q) \in C$  are related by the hyperelliptic involution, and  $\tilde{C}$  is itself hyperelliptic, had been previously studied in detail in many publications, see e.g.,

[20, 25, 13]. Namely, let  $f(x)$  be of odd order and  $x_P = x_Q = x^*$ , where  $x^*$  is not a root of  $f(x)$ . The equation of  $\tilde{C}$  becomes

$$z^2 = x - x^*, \quad y^2 = f(x),$$

and, assuming  $x^* = 0$  without loss of generality, one obtains  $\tilde{C}$ :  $\{y^2 = f(z^2)\}$ . The involution  $\sigma : (z, y) \mapsto (-z, y)$  commutes with the hyperelliptic involution  $\iota : (z, y) \rightarrow (z, -y)$  and yields the second involution  $\tau : (z, y) \mapsto (-z, -y)$  with two fixed (infinite) points on  $\tilde{C}$ . The factor  $\tilde{C}/\tau$  is the second genus  $g$  hyperelliptic curve  $C' : \{v^2 = uf(u)\}$  whose Jacobian is isomorphic to  $\text{Prym}(\tilde{C}, \sigma)$ . In a similar way,  $\text{Prym}(\tilde{C}, \tau)$  is isomorphic to  $\text{Jac}(C)$ . Various applications of this special case of the covering  $\tilde{C} \rightarrow C$  to integrable systems were considered, amongst others, in [12, 13, 18, 23, 24].

One should stress that in the general case of the covering  $\tilde{C} \rightarrow C$ , when  $x_P \neq x_Q$ , there is no second involution on  $\tilde{C}$ , so the latter cannot be a 2-fold covering of the second curve  $C'$ . In particular, this means that meromorphic differentials on  $\tilde{C} \rightarrow C$ , which are anti-invariant with respect to  $\sigma$ , cannot be reduced to differentials on  $C'$ .

## 2.1 Trigonal genus 4 curve and two curves of genus 2

We now apply the above observations to the trigonal curve

$$\tilde{C} : W^3 + azW^2 + bWz^2 + cW + dz^5 + ez^3 + kz = 0, \quad (7)$$

with parameters  $a, b, c, d, e, k \in \mathbb{C}$ , which has genus 4 and admits the involution  $v : (z, W) \mapsto (-z, -W)$  with two fixed points  $\tilde{P} = (0, 0), \tilde{Q} = \infty$ , where  $\infty$  is the only point at infinity on  $\tilde{C}$ . Thus the involution induces the covering  $\pi : \tilde{C} \rightarrow C$  over a hyperelliptic genus two curve  $C$ . Namely, upon introducing the new variable  $x = z/W$ , which is invariant with respect to  $v$ , the equation of  $\tilde{C}$  is rewritten as

$$dz^4x^3 + (1 + ax + bx^2 + ex^3)z^2 + kx^3 + cx^2 = 0.$$

Solving it with respect to  $z^2$ , we get the equation

$$\begin{aligned} z^2 &= \frac{-h(x) + \sqrt{f(x)}}{2dx^3}, \quad h(x) = ex^3 + bx^2 + ax + 1, \\ f(x) &= \alpha x^6 + \beta x^5 + \gamma x^4 + \delta x^3 + \varepsilon x^2 + 2ax + 1, \\ \alpha &= e^2 - 4dk, \quad \beta = 2be - 4dc, \quad \gamma = b^2 + 2ea, \quad \delta = 2e + 2ab, \quad \varepsilon = a^2 + 2b. \end{aligned} \quad (8)$$

Making the birational transformation  $(x, z) \rightarrow \left(x, \frac{z}{\sqrt{2dx}}\right)$ , we finally obtain  $\tilde{C}$  in the “canonical” form (2), that is

$$z^2 = \frac{y - h(x)}{x}, \quad y^2 = f(x) \quad (9)$$

Thus  $\tilde{C}$  is a 2-fold covering of the genus 2 curve  $C : \{y^2 = f(x)\}$  ramified at the points

$$P = \left(x_P = -\frac{c}{k}, \quad y_P = 1 - \frac{ca}{k} + \frac{c^2b}{k^2} - \frac{c^3e}{k^3}\right), \quad Q = (x_Q = 0, \quad y_Q = 1).$$

Note that

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x) = x \cdot (x + c/k) \cdot (2\sqrt{dk}x^2)^2 = 4d(kx + c)x^5, \quad (10)$$

that is,  $\rho(x) = 2\sqrt{dk}x^2$ .

The Prym variety  $\text{Prym}(\tilde{C}, v)$  is principally polarized and isomorphic to the Jacobian of the second genus 2 curve  $C'$ , which, according to Theorem 1, can be written as follows:

$$[w^2(x - x_Q) - h(x)]^2 = h^2(x) - f(x);$$

that is,

$$C' : w^4(x - x_Q)^2 - 2w^2(x - x_Q)h(x) + f(x) = 0. \quad (11)$$

with  $x_Q = 0$  and  $h(x), f(x)$  described in (8). The explicit hyperelliptic form (6) of (11) will be given for some specific values of the parameters  $a, b, c, d, e, k$  in the next section.

### 3 Trigonal spectral curve for generalized Hénon-Heiles systems

Following an observation of A. Fordy [14], it is known that the generalized Hénon-Heiles system with the Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) + c_1 q_1^2 + c_2 q_2^2 + \mathfrak{a} q_1 q_2^2 + \frac{\mathfrak{b}}{3} q_1^3 - \frac{\ell^2}{2q_2^2},$$

that contains an additional non-polynomial term  $\ell^2/(2q_2^2)$  in the potential, is Liouville integrable for 3 sets of the parameters  $c_1, c_2, \mathfrak{a}/\mathfrak{b}$ , namely when the latter ratio takes the values

$$\frac{\mathfrak{a}}{\mathfrak{b}} = \quad (\text{i}) \ 1, \quad (\text{ii}) \ 6, \quad (\text{iii}) \ 16,$$

and  $c_1$  and  $c_2$  are themselves in a suitable ratio in each case. As was done (for  $\ell = 0$ ) in [3], these three integrable cases can be isolated using Painlevé analysis, while Ziglin's theorem can be used to establish non-integrability when the ratio  $\mathfrak{a}/\mathfrak{b}$  does not take one of these values [17].

It was shown in [14, 15] that the integrable cases, numbered (i), (ii), and (iii) above, can be identified with appropriate finite-dimensional reductions of the integrable Sawada-Kotera (SK),  $KdV_5$  and the Kaup-Kupershmidt (KK) PDEs, respectively. This property enabled the construction of matrix Lax representations for each of these three cases.

For  $\ell = 0$ , the general solution for case (ii) was obtained in [27] in terms of hyperelliptic quadratures by a separation of variables in parabolic coordinates in the  $(q_1, q_2)$ -plane (the generalizations of such separable systems were studied in [9]), while the equations of motion for the cases (i) and (iii) were integrated in terms of elliptic functions associated to a pair of different elliptic curves in [4, 21].

Henceforth we concentrate on the algebro-geometric description of the cases (i) and (iii) for  $\ell \neq 0$ .

### 3.1 Case (i): Lax pair, spectral curve and Bäcklund transformation

The Hamiltonian  $H_1(p, q)$  and the additional integral  $H_2(p, q)$  for the case (i) can be written in the form

$$\begin{aligned} H_1 &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{6}q_1^3 + \frac{1}{2}q_1q_2^2 - \frac{\ell^2}{2q_2^2}, \\ H_2 &= H_{20}^2 - \ell^2 \left( \frac{p_1^2}{q_2^2} + \frac{2}{3}q_1 \right), \quad H_{20} = p_1p_2 + \frac{1}{2}q_1^2q_2 + \frac{1}{6}q_2^3. \end{aligned} \quad (12)$$

The integrals commute with respect to the standard Poisson bracket  $\{q_j, p_k\} = \delta_{jk}$  on the phase space  $\mathbb{R}^4$ , and the generic real invariant manifolds of the system are two-dimensional tori.

From [5, 15], the Hamilton's equations corresponding to  $H_1$  admit the following Lax representation with spectral parameter  $\lambda \in \mathbb{C}$ :

$$\dot{L}(\lambda) = [L(\lambda), N(\lambda)], \quad (13)$$

$$L(\lambda) = \begin{pmatrix} 6\lambda q_1 & -\frac{1}{2}(3q_1^2 + q_2^2) & 9\lambda - 3p_1 \\ 9\lambda^2 + 3\lambda p_1 & -3\lambda q_1 - q_2q_2 & q_2^2 \\ -\frac{1}{2}\lambda(3q_1^2 + q_2^2) & 9\lambda^2 + \frac{\ell^2}{q_2^2} - p_2^2 & q_2p_2 - 3\lambda q_1 \end{pmatrix}, \quad N(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -q_1 & 0 \end{pmatrix}; \quad (14)$$

the dot stands for the derivative with respect to time  $t$ .

In addition, the paper [5] presented a family of exact discretizations (Bäcklund transformations)  $\mathcal{B} : (p, q) \rightarrow (\tilde{p}, \tilde{q})$  for the case (i) of the generalized Hénon–Heiles system. The mapping  $\mathcal{B}$  depends on a parameter  $\mu \in \mathbb{C}$  and is described by the intertwining relation

$$\tilde{L}(\lambda)M(\lambda|\mu) = M(\lambda|\mu)L(\lambda), \quad (15)$$

$$M(\lambda|\mu) = \begin{pmatrix} \mu - \lambda & -2Y_2 & 2Y_1 \\ 2Y_1\lambda & 2Y_1Y_2 - \lambda - \mu & -2Y_1^2 \\ 2(Y_2 - 2Y_1^2)\lambda & 2Y_2(Y_2 - 2Y_1^2) + 2Y_1(\lambda + \mu) & -2Y_1(Y_2 - 2Y_1^2) - \lambda - \mu \end{pmatrix}. \quad (16)$$

Here  $L(\lambda)$  is the same as in (14), and  $\tilde{L}(\lambda)$  depends on  $\tilde{p}, \tilde{q}$  in the same way as  $L(\lambda)$  does on  $p, q$ , whereas  $Y_1, Y_2$  are certain functions depending on the old and new variables  $p, q, \tilde{p}, \tilde{q}$ . Thus the mapping  $\mathcal{B}$  is *implicit*. The procedure of its evaluation was described in [5] and will also be sketched in the next section.

The spectral curve preserved by the continuous Hénon–Heiles flow and by the transformation  $\mathcal{B}$  is

$$\mathcal{S} : \det(\eta I - L(\lambda)) \equiv \eta^3 - \ell^2\eta - 729\lambda^5 + 162h_1\lambda^3 - 9h_2\lambda = 0 \quad (17)$$

with  $h_1, h_2$  denoting fixed values of the constants of motion  $H_1, H_2$ .

One sees that  $\mathcal{S}$  is of the type (7) with

$$a = b = 0, \quad c = -\ell^2, \quad d = -729, \quad e = 162h_1, \quad k = -9h_2,$$

so it is trigonal of genus 4, and admits the involution  $v : (\lambda, \eta) \mapsto (-\lambda, -\eta)$  with two fixed points  $(0, 0)$  and  $\infty$ , hence it is a 2-fold covering of a genus 2 curve  $C = \mathcal{S}/v$ . It follows that  $\mathcal{S}$  admits a representation of the form (9) with

$$h(x) = 162 h_1 x^3 + 1, \quad \rho(x) = 162 \sqrt{h_2} x^2, \quad x_P = -\frac{\ell^2}{9h_2}, \quad x_Q = 0.$$

Then, from item c) of Theorem 1, we immediately obtain the following.

**Proposition 2.** *The 4-dimensional Jacobian variety  $\text{Jac}(\mathcal{S})$  contains the 2-dimensional Prym variety  $\text{Prym}(\mathcal{S}, v)$ , which is isomorphic to the Jacobian of the genus 2 curve<sup>3</sup>*

$$\begin{aligned} C' : \quad v^2 &= h_2^3(9u^6 - 27u^4 + 27u^2 - 9) + 2\ell^6(\sqrt{h_2}u + h_1) \\ &\equiv 9h_2^3(u^2 - 1)^3 + 2\ell^6(\sqrt{h_2}u + h_1). \end{aligned} \quad (18)$$

Obviously, the variety  $\text{Jac}(\mathcal{S})$  also contains another Abelian subvariety: the Jacobian of the curve  $C = \mathcal{S}/v$ . As we will see shortly, the complex two-dimensional invariant manifold of the generalized Hénon–Heiles (i) system is an open subset of  $\text{Jac}(C')$  and not of  $\text{Jac}(C)$ .

### 3.2 Cases (iii) and (i): separation of variables and the canonical transformation

In case (iii) the Hamiltonian  $K_1$  and the additional integral  $K_2$  have the form

$$\begin{aligned} K_1 &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{4}{3}q_1^3 + \frac{1}{4}q_1q_2^2 - \frac{2\ell^2}{q_2^2}, \\ K_2^2 &= K_0^2 - \frac{4\ell^2}{3}q_1 - 8\ell^2\frac{p_2^2}{q_2^2} - 4\frac{\ell^2}{q_2^4}, \\ \text{where } K_0^2 &= p_2^4 - \frac{1}{72}p_2^6 - \frac{1}{12}q_1^2q_2^4 + q_1p_2^2q_2^2 - \frac{1}{3}p_1p_2q_2^3. \end{aligned}$$

The paper [26] provided a non-trivial separation of variables for this system, which led to quadratures related to a genus 2 hyperelliptic curve  $\mathcal{K}$ . Using a birational canonical (also non-trivial !) transformation between the cases (i) and (iii), found in [22] for  $\ell = 0$  and in [2, 5] for the general case, these quadratures were also adopted for a linearization of the case (i) of the Hénon–Heiles system. Namely, the canonical transformation implies the following relation between the above integrals  $H_1, H_2$  and  $K_1, K_2$ :

$$K_1 = H_1, \quad K_2^2 = 4H_2.$$

Then, following section V of [26], for the case (i) the coordinates  $q_1, q_2$  read

$$\begin{aligned} q_1 &= \sqrt{-3} \frac{V_1 - V_2}{U_1 - U_2} + U_1^2 + U_1U_2 + U_2^2 + \frac{3h_2}{\ell^2}, \\ q_2^2 &= -2\sqrt{-3}(U_1 + U_2) \frac{U_1V_1 - U_2V_2}{U_1 - U_2} + 2(U_1 + U_2)^2 \left( U_1^2 + U_2^2 + \frac{9h_2}{2\ell^2} \right), \end{aligned} \quad (19)$$

---

<sup>3</sup>Here we omitted a power of the constant factor  $x_P - x_Q = x_P$  in front of  $v^2$ .



where, as above,  $h_1, h_2$  are values of the integrals  $H_1, H_2$ , and  $(U_1, V_1), (U_2, V_2)$  are conjugated separating variables: coordinates of two points on the genus 2 curve

$$\mathcal{K} : \quad V^2 = P_6(U) = 2h_1 - \frac{1}{3} \left( U^2 + \frac{3h_2}{\ell^2} \right)^3 - \frac{2}{3} \sqrt{-3\ell^2} U. \quad (20)$$

A similar rational expressions for  $p_1$  and the product  $q_2 p_2$  in terms of  $(U_1, V_1), (U_2, V_2)$  can be given.

One can observe that  $q_1, q_2^2, p_1, q_2 p_2$  are meromorphic functions on  $\text{Jac}(\mathcal{K})$ , but  $q_2, p_2$  are not. Next, the invariant tori  $\mathcal{I}_h$  of the Hénon–Heiles systems and the systems themselves are invariant with respect to the involution  $\kappa : (q_1, q_2, p_1, p_2) \rightarrow (q_1, -q_2, p_1, -p_2)$ .

The evolution of  $(U_1, V_1), (U_2, V_2)$  with respect to time  $t$  is described by the Abel–Jacobi equations

$$\frac{dU_1}{\sqrt{P_6(U_1)}} + \frac{dU_2}{\sqrt{P_6(U_2)}} = 0, \quad \frac{U_1 dU_1}{\sqrt{P_6(U_1)}} + \frac{U_2 dU_2}{\sqrt{P_6(U_2)}} = dt. \quad (21)$$

It follows that the generalized Hénon–Heiles system (i) is linearized on the Jacobian variety of the curve  $\mathcal{K}$  and the factor of its generic complex invariant manifold  $\mathcal{I}_h$  by  $\kappa$  is an open subset of  $\text{Jac}(\mathcal{K})$ .

In this connection, a natural question is that of how  $\mathcal{K}$  and the genus 4 spectral curve  $\mathcal{S}$  are related. The answer comes out immediately when one observes that the genus 2 curves (20) and (18) are birationally equivalent. Indeed, setting  $U = i\sqrt{3h_2}u/\ell$ , in (20) we get

$$V^2 = 2h_1 - \frac{9h_2^3}{\ell^6} (1 - u^2)^3 + 2\sqrt{h_2} u,$$

and the polynomial in  $u$  differs from the right hand side of (18) only by the factor  $\ell^6$ .

Since the transformation between the Hénon–Heiles cases (i) and (iii) is *birational* and, according to [26], the variables  $q_1, q_2^2, p_1, p_2^2$  for case (iii) are also meromorphic functions on  $\text{Jac}(\mathcal{K})$ , in view of Proposition 2, we arrive at the following

**Proposition 3.** *For generic values of the constants of motion  $h_1, h_2$ , the factorized complex invariant manifold  $\mathcal{I}_h/\kappa$  of the continuous Hénon–Heiles (i) system, as well as of its Bäcklund transformation  $\mathcal{B}$ , is an open subset of the Prym variety  $\text{Prym}(\mathcal{S}, v)$  of the spectral curve  $\mathcal{S}$ .*

**Remark.** The expressions (19) jointly with similar expressions for  $p_1, p_2$  in terms of the separation variables  $(U_1, V_1), (U_2, V_2)$  can be inverted. On the other hand, following known theorems (see e.g., [10]) the evolution of  $U_j, V_j$  by virtue of the Abel equations (21) can be described by means of a  $2 \times 2$  matrix Lax representation with a rational spectral parameter. All this implies that the generalized Hénon–Heiles systems also admit such a  $2 \times 2$  matrix Lax pair.

It remains to give a geometric description of the transformation  $\mathcal{B}$ , as an addition/translation on  $\text{Prym}(\mathcal{S}, v)$  and  $\text{Jac}(\mathcal{K})$ , which will be done in the next section.

## 4 Algebraic geometrical description of the mapping $\mathcal{B}$

Following [5], the mapping  $\mathcal{B}$  is evaluated as follows. In view of (16),

$$\det M(\lambda|\mu) = -(\lambda - \mu)(\lambda + \mu)^2, \quad (22)$$

and  $M(\mu|\mu)$  has a one-dimensional kernel spanned by  $\Phi = (1, Y_1, Y_2)^T$ , whereas the kernel of  $M(-\mu|\mu)$  is the two-dimensional space generated by  $(0, Y_1, Y_2)^T, (Y_2, \mu, 0)^T$ . One can write

$$M(-\mu|\mu) = 2 \begin{pmatrix} 1 \\ -Y_1 \\ -Y_2 + 2Y_1^2 \end{pmatrix} (\mu \quad -Y_2 \quad Y_1). \quad (23)$$

By applying both sides of the Lax relation (15) to  $\Phi$  and setting  $\lambda = \mu$ , we see that  $\Phi$  is an eigenvector of  $L(\mu)$ :

$$L(\mu)\Phi = \eta^*\Phi$$

for a point  $(\mu, \eta^*) \in \mathcal{S}$ . Assume that the parameter  $\mu$  is generic in the sense that the covering  $(\lambda, \eta) \rightarrow \lambda$  is not ramified above  $\lambda = \mu$ . Then, fixing  $\eta^*$  as one of the three possible eigenvalues of  $L(\mu)$ , we determine the values of  $Y_1, Y_2$  in  $\Phi = (1, Y_1, Y_2)^T$  in terms of  $p_i, q_i$  uniquely, and, therefore fix a branch of the transformation  $\mathcal{B}$ . It follows that for a generic  $\mu$  there are three different branches.

Explicitly, we have

$$\begin{aligned} Y_1(\mu, \eta^*) &= \frac{2(9^2\mu^3 - (9p_1^2 + 6q_1q_2^2)\mu + q_2^2\eta^*)}{54\mu^2q_1 + 18(\eta^* - q_1p_1 + q_2p_2)\mu - 6\eta^*p_1 - 6q_2H_{20}}, \\ Y_2(\mu, \eta^*) &= \frac{2\eta^2 + 2q_2p_2\eta - 9(q_1^2 - q_2^2)\mu^2 - (6q_1\eta - 3q_2^2p_1 - 9q_1^2p_1 + 12q_1q_2p_2)\mu}{54\mu^2q_1 + 18(\mu - q_1p_1 + q_2p_2) - 6\mu p_1 - 6q_2H_{20}}, \end{aligned} \quad (24)$$

with  $H_{20}$  being as defined in (12). Substituting these expressions into  $M(\lambda|\mu)$  in (16), from (15) one can evaluate  $\tilde{q}_i, \tilde{p}_i$  following the steps indicated in section 5 of [5].

**The mapping  $\mathcal{B}$  as a translation on  $\text{Prym}(\mathcal{S}, v)$  and  $\text{Jac}(\mathcal{K})$ .** Consider the eigenvector bundle  $\mathbb{P}^2 \rightarrow \mathcal{S}$  associated with the Lax matrix  $L(\lambda)$  in (14),

$$\mathcal{S} \ni P = (\lambda, \eta) \longrightarrow \psi(P) = (\psi_1(P), \psi_2(P), \psi_3(P))^T \quad \text{such that} \quad L(\lambda)\psi(P) = \eta\psi(P),$$

and the equivalence class  $\{\mathcal{D}\}$  of effective divisors of poles of  $\psi(P)$  on the spectral curve  $\mathcal{S}$ . According to the theory (see e.g., [8]) for a generic  $r \times r$  Lax matrix  $L(\lambda)$ , any divisor  $\mathcal{D} \in \{\mathcal{D}\}$  is given by  $n = g + r - 1$  points  $P_1, \dots, P_n$  on the genus  $g$  spectral curve  $\mathcal{S}$ , so in our case  $\deg \mathcal{D} = 6$ . The equivalence class  $\{\mathcal{D}\}$  defines a point in  $\text{Jac}(\mathcal{S})$  via the Abel (Albanese) map

$$\mathcal{D} \mapsto \int_{P_0}^{P_1} \bar{\omega} + \dots + \int_{P_0}^{P_n} \bar{\omega},$$

where  $P_0$  is a base point of the map and  $\bar{\omega}$  is a vector of holomorphic differentials forming a basis in  $H^1(\mathcal{S}, \mathbb{C})$ .

Let now  $\tilde{\psi}(P)$  be a section of the eigenvector bundle associated with  $\tilde{L}(\lambda)$  and  $\{\tilde{\mathcal{D}}\}$  be the corresponding equivalence class. To describe explicitly the relation between  $\mathcal{D}$

and  $\tilde{\mathcal{D}}$ , and, therefore, the corresponding translation on  $\text{Jac}(\mathcal{S})$ , we follow the standard procedure explained, for example, in [25].

First, as follows from the form of the discrete Lax representation (15),

$$\tilde{\psi}(P) = M(\lambda|\mu)\psi(P).$$

Hence,  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  can differ by fixed points  $\mathcal{Q}$  on  $\mathcal{S}$  at which  $M(\lambda, \mu)$  is degenerate ( $M(\lambda|\mu)\psi(\mathcal{Q}) = 0$ ) or its determinant has a pole. In view of (22), the only candidates for such points are

$$\begin{aligned} Q_1 &= (\mu, \eta_1), & Q_2 &= (\mu, \eta_2), & Q_3 &= (\mu, \eta_3), \\ v(Q_1) &= (-\mu, -\eta_1), & v(Q_2) &= (-\mu, -\eta_2), & v(Q_3) &= (-\mu, -\eta_3), \end{aligned}$$

i.e. the points over  $\lambda = \mu$  and  $\lambda = -\mu$  respectively, and the point at infinity on  $\mathcal{S}$ .

**Proposition 4.** *Let us fix a branch of  $\mathcal{B}$  by choosing  $(\mu, \eta^*) = Q_\alpha$ ,  $\alpha \in \{1, 2, 3\}$ . Then*

1) *The relation between  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  is*

$$\tilde{\mathcal{D}} + Q_\alpha + v(Q_\beta) + v(Q_\gamma) \equiv \mathcal{D} + 3\infty, \quad (\alpha, \beta, \gamma) = (1, 2, 3). \quad (25)$$

2) *The shift divisor*

$$\mathcal{U} = \tilde{\mathcal{D}} - \mathcal{D} \equiv Q_\alpha + v(Q_\beta) + v(Q_\gamma) - 3\infty$$

*is equivalent to  $\mathcal{U} \equiv Q_\alpha - v(Q_\alpha)$  and thus defines a point on the subvariety  $\text{Prym}(\mathcal{S}, v) \subset \text{Jac}(\mathcal{S})$ .*

**Corollary.** As follows from item 2) of the above proposition, upon replacing  $Q_\alpha = (\mu, \eta^*)$  by  $v(Q_\alpha) = (-\mu, -\eta^*)$  the shift  $\mathcal{U}$  changes sign. Thus, replacing  $M(\lambda|\mu)$  in the Lax pair (15) by  $M(\lambda|-\mu)$  we get the map  $\tilde{\mathcal{B}}$ , also having 3 branches, which are inverse to those of  $\mathcal{B}$ .

In the special case  $\mu = 0$  one has  $Q_1 = (0, 0)$ ,  $Q_2 = (0, \ell)$ ,  $Q_3 = v(Q_2) = (0, -\ell)$ , so the corresponding branches of  $\mathcal{B}$  are the identity map, and the two others that are inverse to each other. In the latter case expressions (24) simplify to

$$Y_1 = -\frac{q_2^2 \ell}{q_2 H_{20} + p_1 \ell}, \quad Y_2 = \frac{\ell(q_2 p_2 + \ell)}{q_2 H_{20} + p_1 \ell};$$

however, the explicit formulae for the transformation  $(p, q) \rightarrow (\tilde{p}, \tilde{q})$  are still too cumbersome to show here.

*Proof of Proposition 4.* 1) Let  $M^*(\lambda|\mu)$  denote the matrix  $M(\lambda|\mu)$  with the values of  $Y_1, Y_2$  chosen above. Then we have

$$\begin{aligned} M^*(\mu|\mu) \psi(Q_\alpha) &= 0, & M^*(\mu|\mu) \psi(Q_\beta) &\neq 0, & M^*(\mu|\mu) \psi(Q_\gamma) &\neq 0, \\ M^*(-\mu|\mu) \psi(v(Q_\alpha)) &\neq 0, & M^*(-\mu|\mu) \psi(v(Q_\beta)) &= 0, & M^*(-\mu|\mu) \psi(v(Q_\gamma)) &= 0. \end{aligned} \quad (26)$$

Indeed,  $\psi(Q_\alpha)$  is proportional to  $(1, Y_1, Y_2)^T$ , the kernel of  $M^*(\mu|\mu)$ . The latter is one-dimensional, which implies the first line in (26).

Next, by construction, the vectors  $\psi(v(Q_\beta)), \psi(v(Q_\gamma))$  can be written explicitly as

$$(1, Y_1(-\mu, -\eta_\beta), Y_2(-\mu, -\eta_\beta))^T, \quad \text{respectively} \quad (1, Y_1(-\mu, -\eta_\gamma), Y_2(-\mu, -\eta_\gamma))^T,$$

where  $Y_1(-\mu, -\eta_\beta), Y_2(-\mu, -\eta_\beta)$  are evaluated as in (24). Due to (23),  $M^*(-\mu, \mu) \psi(v(Q_\beta))$  is proportional to

$$\begin{aligned} & \Pi_\beta(1, -Y_1, -Y_2 + 2Y_2^2)^T, \\ \Pi_\beta = & \langle (\mu, -Y_2(\mu, \eta^*), Y_1(\mu, \eta^*))^T, (1, Y_1(-\mu, -\eta_\beta), Y_2(-\mu, -\eta_\beta))^T \rangle. \end{aligned}$$

Explicitly, up to a constant factor, the above scalar product reads

$$\begin{aligned} \Pi_\beta = & q_2^2 [729\mu^5 - 27(3p_1^2 + 3p_2^2 + q_1^3 + 3q_1q_2)\mu^3 + 9H_{20}^2\mu] \\ & + ((\eta^*)^2 + \eta^*\eta_\beta + \eta_\beta^2)[9^2\mu^3 - 3(2q_1q_2^2 + 3p_1^2)\mu] + \eta^*\eta_\beta(\eta^* + \eta_\beta)q_2^2. \end{aligned} \quad (27)$$

Due to the form of the spectral curve  $\mathcal{S}$  in (17),

$$\eta_\alpha + \eta_\beta + \eta_\gamma = 0, \quad \eta_\alpha\eta_\beta + \eta_\gamma\eta_\alpha + \eta_\beta\eta_\gamma = -\ell^2,$$

which implies

$$(\eta^*)^2 + \eta^*\eta_\beta + \eta_\beta^2 = \ell^2, \quad \eta^*\eta_\beta(\eta^* + \eta_\beta) = -\eta^*\eta_\beta\eta_\gamma = -\det L(\mu).$$

Then the right hand side of (27) can be written as

$$q_2^2 [729\mu^5 - 162h_1\mu^3 + 9h_2\mu - \det L(\mu)],$$

which is zero in view of the equation of  $\mathcal{S}$ . Hence  $M^*(-\mu, \mu) \psi(v(Q_\beta)) = 0$ .

The same argument shows that  $M^*(-\mu, \mu) \psi(v(Q_\gamma)) = 0$ . Since  $\psi(v(Q_\beta)), \psi(v(Q_\gamma))$  span the whole kernel of  $M^*(-\mu, \mu)$ , we conclude that  $M^*(-\mu, \mu) \psi(v(Q_\alpha)) \neq 0$ . Thus, the inverse operator  $(M^*(\lambda, \mu))^{-1}$  acting on  $\tilde{\psi}(P)$ ,  $P \in \mathcal{S}$  produces poles of  $\psi(P)$  only at  $Q_\alpha, v(Q_\beta), v(Q_\gamma)$ . Therefore, to get  $\mathcal{D}$ , one must add the above points to  $\tilde{\mathcal{D}}$ . On the other hand,  $\tilde{\mathcal{D}}$  is obtained from  $\mathcal{D}$  by adding a multiple of  $\infty \in \mathcal{S}$ . Since the degree of  $\tilde{\mathcal{D}} - \mathcal{D}$  must be zero, we obtain the relation (25).

2) Adding to  $\mathcal{U}$  the divisor of the meromorphic function  $1/(\lambda + \mu)$ , that is

$$3\infty - v(Q_1) - v(Q_2) - v(Q_3),$$

we get  $Q_\alpha - v(Q_\alpha)$ , which is obviously anti-invariant with respect to the involution  $v$ . Thus  $\mathcal{U}$  defines a point on the Prym variety, by the definition of the latter.  $\square$

**Remark.** Since  $\text{Prym}(\mathcal{S}, v)$  is identified with the Jacobian of the curve  $\mathcal{K}$  associated with the separation of variables, it is natural to describe the shift  $Q_\alpha - v(Q_\alpha)$  on Prym explicitly as an equivalence class  $\{\mathcal{P}_1 - \mathcal{P}_2\}$ , where  $\mathcal{P}_1, \mathcal{P}_2$  are effective divisors of equal degree on  $\mathcal{K}$ . This can be made by considering Jacobians of the curves which form the corresponding tower (tree) of curves given in [7]. We want to avoid this specific analysis in the present paper, so an explicit derivation of  $\mathcal{P}_1, \mathcal{P}_2$  will be made in another publication in a broader context of hyperelliptic Prym varieties.

## Acknowledgments

The authors are grateful to Aaron Levin for valuable comments concerning Theorem 1 and to Harry Braden for stimulating discussions. The work of V.E was supported by the School of Mathematics, University of Edinburgh, with the certificate of sponsorship C5E7V94128U. Yu.F acknowledges the support of the MICIIN grants MTM2012-31714 and MTM2012-37070.

## References

- [1] Adler, M., van Moerbeke, P. The Kowalewski and Hénon-Heiles motions as Manakov geodesic flows on  $SO(4)$ : a two-dimensional family of Lax pairs. *Comm. Math. Phys.* **113** (1988), no. 4, 659–700
- [2] Blaszak, M., Rauch–Wojciechowski, S. Soliton point particles of extended evolution equation. *J. Math. Phys.* **35** (1994) 1693–1709.
- [3] Chang, Y.F., Tabor, M. and Weiss, J. Analytic structure of the Hénon-Heiles Hamiltonian in integrable and nonintegrable regimes. *J. Math. Phys.* **23** (1982) 531–538.
- [4] Chazy, J. Thesé. Paris, 1910; *Acta Math.* **34**, 317–1911
- [5] Common, Alan K.; Hone, Andrew N. W.; Musette, Micheline, A new discrete Hénon–Heiles system. *J. Nonlinear Math. Phys.* **10** (2003), suppl. 2, 27–40.
- [6] Cosgrove, Christopher M. Higher-order Painlevé equations in the polynomial class. I. Bureau symbol P2. *Stud. Appl. Math.* **104** (2000), no. 1, 1–65
- [7] Dalaljan, S.G. The Prym variety of a two-sheeted covering of a hyperelliptic curve with two branch points, (Russian) *Mat. Sb. (N.S.)*, **98** (140) (1975), no. 2 (10), 255–267, 334.
- [8] Dubrovin B.A., Novikov S.P., Krichever I.M.. *Integrable Systems. I. Itogi Nauki i Tekhniki. Sovr.Probl.Mat. Fund.Naprav.* Vol.4, VINITI, Moscow 1985. English transl.: *Encyclopaedia of Math.Sciences*, Vol. 4, Springer-Verlag, Berlin 1989
- [9] Eilbeck J.C., Enolskii V.Z., Kuznetsov V.B., Leykin D.V. Linear  $r$ -matrix algebra for the systems separable in parabolic coordinates, *Phys. Lett.*, **180A**, no. 3, (1993) 208–214
- [10] Fairbanks L. Lax equation representation of certain completely, integrable systems. *Comp.Math.* **68**, No.1 (1988), 31–40
- [11] Fay, J. D. *Theta functions on Riemann surfaces*, Lectures Notes in Mathematics (Berlin), Vol. 352, Springer, 1973.
- [12] Fernandes, R., Santos, J. Integrability of the periodic KM system. *Rep. Math. Phys.* **40** (1997), 475–484.

- [13] Fernandes, R. L.; Vanhaecke, P. Hyperelliptic Prym varieties and integrable systems. *Comm. Math. Phys.* **221** (2001), no. 1, 169–196
- [14] Fordy, A. P. The Hénon-Heiles system revisited. *Phys. D* **52** (1991), no. 2-3, 204–210.
- [15] Fordy, A. P. Stationary flows: Hamiltonian structures and canonical transformations. *Phys. D* **87** (1995), no. 1-4, 20–31
- [16] Horozov E., van Moerbeke, P. The full geometry of Kowalewski’s top and  $(1, 2)$ -abelian surfaces. *Comm. Pure Appl. Math.* **42**:4 (1989) 357–407.
- [17] Ito, H. Non-integrability of Hénon-Heiles system and a theorem of Ziglin. *Kodai Math. J.* **8** (1985) 120–138
- [18] Kuznetsov, V. Separation of variables for the  $D_n$ -type periodic Toda lattice. *J. Phys. A: Math. and Gen.* **30** (1997) 2127
- [19] Levin, A. Siegel’s theorem and the Shafarevich conjecture. *J. Théor. Nombres Bordeaux* **24** (2012), no. 3, 705–727
- [20] Mumford. D. Prym Varieties I. in: *Contributions to analysis*, Ahlfors L.V. Kra I. Maskit B. Nirenberg L., Eds., Academic Press (1974), 325–350.
- [21] Ravoson, V.; Gavrilov, L.; Caboz, R. Separability and Lax pairs for Hénon-Heiles system. *J. Math. Phys.* **34** (1993), no. 6, 2385–2393
- [22] Salerno, M.; Enolskii, V. Z.; Leykin, D. V. Canonical transformation between integrable Hénon-Heiles systems. *Phys. Rev. E* (3) **49** (1994), no. 6, part B, 5897–5899
- [23] Taimanov, I. Prym varieties of branched coverings, and nonlinear equations. (Russian) *Mat. Sb.* **181** (1990), no. 7, 934–950; translation in *Math. USSR-Sb.* **70** (1991), no. 2, 367–384
- [24] Taimanov, I. Prym theta functions and hierarchies of nonlinear equations. (Russian) *Mat. Zametki* **50** (1991), no. 1, 98–107, 160; translation in *Math. Notes* **50** (1991), no. 1-2, 723–730 (1992)
- [25] van Moerbeke P, Mumford. D. The spectrum of difference operator and algebraic curves. *Acta Math.* (1979)
- [26] Verhoeven, C.; Musette, M.; Conte, R. Integration of a generalized Hénon-Heiles Hamiltonian. *J. Math. Phys.* **43** (2002), no. 4, 1906–1915
- [27] Wojciechowski, S. Separability of an integrable case of the Hénon-Heiles system. *Phys. Lett. A* **100** (1984), no. 6, 277–278
- [28] Lihua Wu, Guoliang He, and Xianguo Geng. Algebro-geometric solutions to the modified Sawada–Kotera hierarchy *J. Math. Phys.*, **53**, (2012) 123–513